Embedding of Coxeter groups in a product of trees

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Abstract

We prove that a right angled Coxeter group Γ with chromatic number n can be embedded in a bilipschitz way into the product of n locally finite trees. We give applications of this result to various embedding problems and determine the hyperbolic rank of products of exponentially branching trees.

1 Introduction

We consider a finitely generated right angled Coxeter group Γ , i.e. a group Γ together with a finite set of generators S, such that every element of S has order two and that all relations in Γ are consequences of relations of the form $st = ts$, where $s, t \in S$.

We prove embedding results of the Cayley graph $C(\Gamma, S)$ into products of trees. On graphs and trees we consider always the simplicial metric, hence every edge has length 1. On a product of trees we consider the l_1-product metric, i.e. the distance is equal to the sum of the distances in the factors.

In [DJ] it was shown that the Cayley graph of a Coxeter group admits an equivariant isometric embedding into a finite product of locally infinite trees. Here we give a better estimate on the number of factors in the right-angled case. The estimate is given in terms of the chromatic number. Consider therefore *colourings* $c : S \to \{1, \ldots, n\}$ with the property that for different $s, t \in S$ with $st = ts$ we have $c(s) \neq c(t)$. The minimal number n of colours needed is called the chromatic number of Γ.

Theorem 1.1. *Suppose that the chromatic number of a right-angled Coxeter group Γ is n. Then the Cayley graph $C(\Gamma, S)$ admits an equivariant isometric embedding into the product of n simplicial trees.*

Besides of trivial cases, these trees are locally infinite. However we are able to embed the Cayley graph bilipschitz into a product of locally compact trees.

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Definition 1.2. *A pointed simplicial tree* (T, t_0) *is called exponentially branching, if there exists a number* $\sigma > 0$ *such that every vertex* $t \in T$ *has more than* $e^{\sigma d(t,t_0)}$ *neighbours where* d *is the metric on* T.

Theorem 1.3. *Let* Γ *be a right-angled Coxeter group with chromatic number* n*, let* T *be an exponentially branching locally compact simplicial tree, and let* $r > 0$ *be a number. Then there exists bilipschitz embedding* $\psi : C(\Gamma, S) \to T \times \cdots \times T$ *(*n*-factors), such that* ψ *restricted to every ball of radius* r *is isometric.*

It is an interesting open problem, if a corresponding embedding result holds for trees with bounded valence.

We can apply Theorem 1.3 for a special Coxeter group operating on the hyperbolic plane $\mathbb{H}^2$ and obtain:

Corollary 1.4. *For every exponentrially branching tree* T *there exists a bilipschitz embedding* $\varphi : \mathbb{H}^2 \to T \times T$.

Combining this with a result of Brady and Farb we get the following higher dimensional version:

Corollary 1.5. *For every exponentrially branching tree* T *there exists a bilipschitz embedding* $\psi : \mathbb{H}^n \to T \times \cdots \times T$ *of the hyperbolic space* $\mathbb{H}^n$ *into the* $2(n-1)$ *fold product of* T.

It is an open question, if for $n \geq 3$ there is a bilipschitz embedding of $\mathbb{H}^n$ into the n-fold product of locally compact trees. There are two partial results in this direction. In [BS2] it is show that there exists a quasiisometric embedding of $\mathbb{H}^n$ into an n-fold product of locally infinite trees. On the other hand a recent construction of Januszkiewicz and Swiatkowski [JS] shows for every n the existence of a right angled Gromov hyperbolic Coxeter group with virtual cohomological dimension and colouring number equal to n. Combining Theorem 1.3 with that result we obtain:

Corollary 1.6. *For every exponentially branching tree* T *and any given number* n *there exits a Gromov hyperbolic group* Γ_n *with virtual cohomological dimension* n *and a bilipschitz embedding of the Cayley graph of* Γ_n *into the product* $T \times \cdots \times T$ *(*n*-factors).*

Corollary 1.6 can be used to determine the hyperbolic rank (compare [BS1]) of a product of trees:

Corollary 1.7. *The hyperbolic rank of the product of* n *trees with exponential branching is* $(n-1)$.

2 Right Angled Coxeter Groups

In this section we review the necessary facts from the theory of right angled Coxeter groups.

2.1 Deletion Rule and Cayley Graph

A Coxeter matrix $(m_{s,t})_{s,t\in S}$ is a symmetric $S \times S$ matrix with 1 on the diagonal and with all other entries nonnegative integers different from 1. A Coxeter matrix defines a Coxeter group Γ generated by the index set S with relations $(st)^{m_{s,t}} = 1$ for all $s, t \in S$. Here we use the convention that $\gamma^0 = 1$ for all elements, thus if $m_{s,t} = 0$ then there is no relation between s and t. A Coxeter group Γ is *finitely generated*, if S is finite. The group Γ is called *right angled*, if all entries of the corresponding Coxeter matrix are 0,1,2. The Coxeter matrix of a right angled Coxeter group is completely described by a graph with vertex set S where we connect two vertices s and t iff $m(s,t) = 2$.

Right angled Coxeter groups have a very simple deletion law. By the following two operations every word w in the generators S can be transformed to a reduced word and two reduced words representing the same element can be transformed by means only the second operation [Br] :

(i) delete a subword of the form ss, $s \in S$

(ii) replace a subword st by ts if $m_{s,t} = 2$.

This deletion rule has the following consequences:

Lemma 2.1. **(a)** *If w and w' represent the same element then the lenghts of w and w' are either both even or both odd.*

(b) *Let w and w' be reduced representations of the same element $\gamma \in \Gamma$, then w and w' are formed from the same set of letters and they have the same length.*

We now investigate some properties of the Cayley graph $C(\Gamma, S)$ of Γ with respect to the generating set S for a right angled Coxeter group. Let $\gamma \in \Gamma$ and let w be a reduced representation of γ. The length of w is denoted by $\ell(\gamma)$ and called the norm of γ. This is well defined by (b). On Γ we consider the distance function $d(\gamma, \beta) = \ell(\gamma^{-1}\beta)$. Let γ and β be elements of Γ which are neighbours in the Cayley graph and let w be a reduced word representing γ. Then there exists a generator $s \in S$ such that β has the representation ws. It follows from (b) that $\ell(\gamma) \neq \ell(\beta)$. Thus an edge in the Cayley graph connects two elements with different norm. This allows us to orient the edges of the Cayley graph and we choose the orientation in the way that the initial point of an edge is closer to the 1-element than the endpoint of the edge. We write an oriented edge as $[\alpha, \beta]$. As usual one can define geodesics in the Cayley graph. A geodesic between two points $\alpha, \beta \in \Gamma$ is given by a sequence $\alpha = \gamma_0, \ldots, \gamma_k = \beta$ with $d(\gamma_i, \gamma_j) = \mid i - j \mid$.

Let α, β and γ be elements of Γ. We say that γ *lies between α and β* if $d(\alpha, \gamma) + d(\gamma, \beta) = d(\alpha, \beta)$. The Cayley graphs of right angled Coxeter

groups have the following property, which says that any three points in Γ span a tripoid:

Lemma 2.2. *Let $\alpha, \beta, \gamma \in \Gamma$, then there exists $\delta \in \Gamma$ such that δ lies between γ_i and γ_j for any choice of distinct elements $\gamma_i, \gamma_j \in \{\alpha, \beta, \gamma\}$.*

Proof. By the Γ-invariance of the metric it suffices to show this result for the case that $\gamma = 1$. Let $\alpha, \beta \in \Gamma$ and consider a geodesic path $\alpha = \alpha_0, \ldots, \alpha_k = \beta$ from α to β and consider the sequence of norms $n_0 = \ell(\alpha_0), \ldots, n_k = \ell(\alpha_k)$. Note that by the properties discussed above $\mid n_i - n_{i+1} \mid = 1$. Assume that there is a subsequence $\alpha_{i-1}, \alpha_i, \alpha_{i+1}$ with $n_{i-1} < n_i > n_{i+1}$. Then one can represent α_i in a reduced way as $w_1 t = w_2 s$ where w_1 is a reduced word representing α_{i-1} and w_2 a reduced word representing α_{i+1} and $s, t \in S$. Since by the deletion law one can transform $w_1 t$ into $w_2 s$ by means of operations of type (ii), we see that $st = ts$ and that one can represent α_i by a reduced word of the form $wst = wts$ where ws represents α_{i-1} and wt represents α_{i+1}. Replace now α_i by the element α_i' represented by w and we obtain a new geodesic sequence between α and β such that for the corresponding sequence of norms we have $n_{i-1} > n_i' < n_{i+1}$. Applying this procedure several times we obtain a geodesic path from α to β, such that the sequence of the norms has no local maximum any more and hence only a global minimum n_{i_0}. The corresponding element $\delta = \alpha_{i_0}$ lies between α and β but also between 1 and α resp. 1 and β.

□

Let $s \in S$ be a generator. We define $Z_s(S) = \{t \in S \mid st = ts\}$ and $Z_s^0(S) = Z_s(S) \setminus \{s\}$. By $Z_s(\Gamma)$ we denote the centralizer of s (in Γ).

Lemma 2.3. *For a right-angled Coxeter group the centralizer $Z_s(\Gamma)$ of a generator $s \in S$ is the subgroup generated by the set $Z_s(S) \subset S$.*

Proof. Clearly all elements in $Z_s(S)$ commute with s. Suppose that w commutes with s. Then $wsw^{-1} = s$. Let $w = s_1 \ldots s_k$ be a reduced presentation. Then $s_1 \ldots s_k s s_k \ldots s_1 = s$. Note that s_k commutes with s. If it does not commute, the letter s in the middle cannot be canceled and hence after every transformation there always will be a letter s_k to the left from s. This contradicts that the length of the word is one. If s_k commutes with s it can be canceled and we consider s_{k-1}. By induction we have that all s_i commute with s. Therefore $s_i \in Z_s(S)$ for $i = 1, \ldots, k$.

□

If w is a reduced representation of an element in $Z_s(\Gamma)$ then all letters of w commute with s and hence the letter s can only occur once. It follows that $Z_s(\Gamma)$ spits in a natural way as $Z_s^0(\Gamma) \times \mathbb{Z}_2$, where $Z_s^0(\Gamma)$ is the subgroup of Γ generated by $Z_s^0(S)$. Note that $Z_s^0(\Gamma)$ is itself a right angled Coxeter group with generating set $Z_s^0(S)$.

Lemma 2.4. *Let $s_1 \dots s_m$ be a reduced word and $s \in S$. Then the following equivalence holds: $ss_1 \dots s_m$ is not reduced $\Leftrightarrow \exists\ i \in \{1, \dots, m\}$ such that $s_i = s$ and $s_1, \dots, s_{i-1} \in Z_s^0(S)$.*

Proof. " $\Leftarrow$ " is clear.

" $\Rightarrow$ " By assumption the word $ss_1 \dots s_m$ is not reduced, let $s'_1, \dots, s'_l$ be a reduced representation of $ss_1 \dots s_m$. It follows from Lemma 2.1 that $l = m - r$ where r is a positive odd number. Since $ss'_1 \dots s'_l$ is a presentation of $s_1 \dots s_m$ we have $l + 1 \geq m$ and thus $l = m - 1$. Hence $ss'_1 \cdots s'_l$ is a minimal representation of $s_1 \dots s_m$. By Lemma 2.1 (b) the letter s occurs in $\{s_1 \dots s_m\}$. Let i be the smallest integer such that $s_i = s$. Assume that $i \geq 2$ and s_1 not in $Z_s^0(S)$. Then the letter s in front of $s_1 \dots s_m$ can not be cancelled with another letter s in $s_1 \dots s_m$. Since $s_1 \dots s_m$ is reduced this implies that also $ss_1 \dots s_m$ is reduced. This contradiction shows that $s_1 \in Z_s^0(S)$. Inductively it follows that $s_1, \dots, s_{i-1} \in Z_s^0(S)$.

□

Lemma 2.5. *Let $s, t \in S$ and let $s_1 \dots s_m$ be a reduced word, such that the words $s_1 \dots s_m t$ and $ss_1 \dots s_m$ are reduced, but the word $ss_1 \dots s_m t$ is not reduced. Then $s = t$ and $s_1, \dots, s_m \in Z_s^0(S)$.*

Proof. Define $s_{m+1} := t$. Since $s_1 \dots s_m s_{m+1}$ is reduced and $ss_1 \dots s_m s_{m+1}$ is not reduced, it follows from Lemma 2.4 that there exists $i \in \{1, \dots, m+1\}$ such that $s_i = s$ and $s_1, \dots, s_i \in Z_s^0(S)$. Since $s_1 \dots s_m$ is reduced and $ss_1 \dots s_m$ is also reduced, it follows from Lemma 2.4 that $i = m + 1$ which implies that $s = t$ and $s_1, \dots, s_m \in Z_s^0(S)$.

□

For a generator $s \in S$ we define the *halfspace* $H_s = \{\gamma \in \Gamma \mid \ell(s\gamma) > \ell(\gamma)\}$ with the boundary $\partial H_s = \{\gamma \in H_s \mid d(\gamma, \Gamma \setminus H_s) = 1\}$. We have the following properties

Lemma 2.6. **(a)** *$\Gamma = H_s \cup sH_s$ is a disjoint union.*

(b) *If $\alpha \in H_s$ and $\alpha t \in sH_s$ for some $t \in S$. Then $\ell(\alpha) < \ell(\alpha t)$, $t = s$ and $\alpha \in Z_s^0(\Gamma)$.*

(c) *$\partial H_s = Z_s^0(\Gamma)$ and H_s is $Z_s^0(\Gamma)$-invariant.*

(d) *H_s and ∂H_s are totally convex, i.e. every geodesic with initial and endpoint in H_s (resp. in ∂H_s) is completely contained in H_s (resp. in ∂H_s).*

Proof. (a) Since trivially $\ell(s\gamma) > \ell(\gamma)$ if and only if $\ell(ss\gamma) < \ell(s\gamma)$ we have $H_s \cap sH_s = \emptyset$. For given $\gamma \in \Gamma$ we know that $\ell(\gamma) \neq \ell(s\gamma)$. Thus $\gamma \in H_s$ or $s\gamma \in H_s$ which implies $\Gamma = H_s \cup sH_s$.

(b) Assume to the contrary that $\ell(\alpha) > \ell(\alpha t)$. Let $s_1 \dots s_k$ be a reduced representation of αt, then $s_1 \dots s_k t$ is a reduced representation of α. Since $ss_1 \dots s_k$ is not reduced it follows trivially that $ss_1 \dots s_k t$ is also not reduced. This is a contradiction to $\alpha \in H_s$. Thus $\ell(\alpha) < \ell(\alpha t)$.

Let now $s_1 \dots s_k$ be a reduced representation of α then $s_1 \dots s_k t$ is a reduced representation of αt. By assumption $ss_1 \dots s_k$ is reduced and $ss_1 \dots s_k t$ is not reduced. Then $t = s$ and $\alpha \in Z_s^0(\Gamma)$ by Lemma 2.4.

(c) $\partial H_s \subset Z_s^0(\Gamma)$ follows immediately from (b). If $s_1 \dots s_k$ is a reduced representation of an element $\gamma \in Z_s^0(\Gamma)$ with $s_i \in Z_s^0(S)$, then $ss_1 \dots s_k$ is a reduced word, i.e. $Z_s^0(\Gamma) \subset H_s$. Clearly $ss_1 \dots s_k \in sH_s$ thus $Z_s^0(\Gamma) \subset \partial H_s$.

To show that H_s is $Z_s^0(\Gamma)$-invariant, we first prove that H_s is star shaped with respect to 1, i.e. every geodesic from 1 to $\gamma \in H_s$ is completely contained in H_s. Such a geodesic corresponds to a reduced representation $s_1 \dots s_m$ of γ. Since $\gamma \in H_s$ we see that $ss_1 \dots s_m$ is a reduced word and hence $ss_1 \dots s_k$ is reduced for all $0 \leq k \leq m$ which implies that the geodesic lies in H_s.

Let now $\gamma \in H_s$ and $\alpha \in Z_s^0(\Gamma)$. Let $1 = \beta_0, \dots, \beta_m = \gamma$ be a geodesic which is by the above completely contained in H_s. Then $\alpha\beta_0, \dots, \alpha\beta_m$ is a geodesic from $\alpha \in H_s$ to $\alpha\gamma$. If $\alpha\gamma$ is not contained in H_s, then this geodesic leaves H_s and there is i such that $\alpha\beta_i \in H_s$ and $\alpha\beta_{i+1} \in sH_s$. By (b) $\alpha\beta_i \in Z_s^0(\Gamma)$ and $\alpha\beta_{i+1} \in Z_s(\Gamma) \setminus Z_s^0(\Gamma)$. Since $\alpha \in Z_s^0(\Gamma)$ this implies $\beta_{i+1} \in (Z_s(\Gamma) \setminus Z_s^0(\Gamma)) \subset sH_s$, a contradiction.

(d) Let $\alpha_0, \dots, \alpha_m$ be a geodesic with initial and endpoint in H_s. If this geodesic is not completely contained in H_s let i be the smallest index such that $\alpha_i \in sH_s$ and j be the largest index with $\alpha_j \in sH_s$. By (b) $\alpha_{i-1}, \alpha_{j+1} \in Z_s^0(\Gamma)$. Thus it remains to show that every geodesic joining two points $\alpha, \beta \in Z_s^0(\Gamma)$ is completely contained in H_s. A geodesic from 1 to $\alpha^{-1}\beta \in Z_s^0(\Gamma)$ corresponds to a reduced word representing $\alpha^{-1}\beta$. By the deletion rule such a word is formed only out of letters from $Z_s^0(S)$. Thus every geodesic from 1 to $\alpha^{-1}\beta$ is contained in $Z_s^0(\Gamma) \subset H_s$ and since H_s is α-invariant by (c) every geodesic from α to β is contained in H_s. The argument also shows that every geodesic between two points $\alpha, \beta \in Z_s^0(\Gamma) = \partial H_s$ is conpletely contained in ∂H_s.

□

Remark 2.7. Lemma 2.6 says that the boundary of $\partial H_s = \{\gamma \in H_s, d(\gamma, \Gamma \setminus H_s) = 1$ is equal to $Z_s^0(\Gamma)$ and $\partial sH_s = sZ_s^0(\Gamma)$.

2.2 Nerve and Davis Complex

Let Γ be a right angled Coxeter group with generating set S. The nerve $N = N(\Gamma, S)$ is the simplicial complex defined in the following way: the vertices of N are the elements of S. Two different vertices s, t are joined by an edge, if and only if $m(s,t) = 2$. In general (k+1) different vertices $s_1, \dots, s_{k+1}$

span a k-simplex, if and only $m(s_i, s_k) = 2$ for all pairs of different $i, j \in \{1, \ldots, , k+1\}$. For a simplex σ of N, let Γ_σ be the subgroup of Γ generated by the vertices of σ. If σ is a k-simplex spanned by $s_1, \ldots, s_{k+1} \in S$ then Γ_σ is isomorphic to $\mathbb{Z}_2^{k+1}$. By N' we denote the barycentric subdivision of N. The cone $C = \operatorname{Cone} N'$ over N' is called a *chamber* for Γ. The Davis complex [D] $X = X(\Gamma, S)$ is the image of a simplicial map $q : \Gamma \times C \to X$ defined by the following equivalence relation on the vertices: $a \times v_\sigma \sim b \times v_\sigma$ provided $a^{-1}b \in \Gamma_\sigma$. Here σ is a simplex in N, Γ_σ is the subgroup of Γ generated by the vertices of σ, v_σ is the barycenter of σ. We identify C with the image $q(1 \times C)$ as a subset of X. The group Γ acts simplicially on X by $\gamma q(\alpha \times x) = q(\gamma\alpha \times x)$ and the orbit space is equal to the chamber C. Thus the Davis complex X is obtained by gluing the chambers γC , $\gamma \in \Gamma$ along the boundaries. Note that X admits an equivariant cell structure with the vertices $X^{(0)}$ equal the cone points of the chambers and with the 1-skeleton $X^{(1)}$ isomorphic to the Cayley graph of Γ.

An alternative description of the Davis complex is obtained in the following way: Consider the cubical cell complex $\widehat{X}$, whose vertex set consists of the elements of Γ. The 1-cells are of the form $[\gamma, \gamma s]$ where $\gamma \in \Gamma$ and $s \in S$. Thus the 1-skeleton $\widehat{X}^{(1)}$ is the Cayley graph $C(\Gamma, S)$. The 2-cells are squares with a vertex set of the form $\gamma, \gamma s, \gamma t, \gamma st = \gamma ts$, where $s, t \in S$ are distinct commuting elements. In general the k-cells are cubes with vertex set $\gamma\Gamma_\sigma$, where σ is a $(k-1)$-simplex in N. Then $\widehat{X}$ can be considered as a cubical realization of the Davis complex.

The generators $s \in S$ and their conjugates $r = \gamma s \gamma^{-1}$, $\gamma \in \Gamma$ are called *reflections.* A *mirror (or wall)* of a reflection $r \in \Gamma$ is the set of fixed points $M_r \subset X$ of r acting on the Davis complex X. Note that $M_{\gamma s \gamma^{-1}} = \gamma M_s$.

Lemma 2.8. *For every generator s in a right-angled Coxeter group Γ there is the equality $M_s = \{q(w \times x) \mid w \in Z_s(\Gamma), x \in St(s, N')\}$.*

Proof. " $\supset$ " : If $w \in Z_s(\Gamma)$ and $x \in St(s, N')$, i.e. x is an affine combination $x = \sum_{s \in \sigma} x_\sigma v_\sigma$ one easily computes

$$sq(w \times x) = wq(s \times x) = wq(1 \times x) = q(w \times x)$$

.
" $\subset$ " : Let $z \in M_s$. Then $z = q(g \times x)$ for some $g \in \Gamma$ and $x \in cone(N')$. The condition $s(z) = z$ can be rewritten as $q(sg \times x) = q(g \times x)$. Hence $g^{-1}sg \in \Gamma_\sigma$ for some simplex σ of N and $x \in \cap_{v \in \sigma} St(v, N')$. By the deletion law $s \in \Gamma_\sigma$, since the number of s in $g^{-1}sg$ is odd. Hence $s \in \sigma$ and $x \in St(s, N')$.

Let $s_1 \ldots s_k$ be a reduced presentation of $g^{-1}sg$. We note that all $s_i \in \sigma$. Since the group is rightangled, Γ_σ is commutative and hence all s_i are different. Let $u_1 \ldots u_l$ be a reduced presentation of g. Note that $s_j \neq u_i$ for every $u_i \neq s$, since u_i appears even number times in the word $u_l \ldots u_1 s u_1 \ldots u_l$. Hence $g^{-1}sg = s$, i.e. $g \in Z_s(\Gamma)$. □

The *chromatic number* of a graph is the minimal number of colours needed to colour the vertices in such a way that every adjacent vertices have different colours. A chromatic number of a simplicial complex is the chromatic number of its 1-dimensional skeleton.

Assume that the chromatic number of the nerve $N(\Gamma)$ of a right-angled Coxeter group Γ equals n and let $c : N^{(0)} \to \{1, \dots, n\}$ be a corresponding colouring map. Then for every mirror gM_s in X we can assign a colour by taking the colour $c(s)$. Similarly we colour every edge $[\gamma, \gamma s]$ of the Cayley graph $C(\Gamma, S)$ with the colour $c(s)$.

Lemma 2.9. *Different mirrors of the same colour are disjoint.*

Proof. Let $c(s) = c(t),\ s, t \in S$ and let $\gamma_1(M_s) \cap \gamma_2(M_t) \neq \emptyset$. Therefore $gM_s \cap M_t \neq \emptyset$ where $g = \gamma_2^{-1}\gamma_1$. Let $x \in gM_s \cap M_t$. By Lemma 2.8 we have $x = q(w \times y) = q(gu \times z)$, where $w \in Z_t(\Gamma)$, $y \in St(t, N')$ and $u \in Z_s(\Gamma)$, $z \in St(s, N')$. Hence $y = z = \sum_{t,s\in\Sigma} y_\sigma v_\sigma$. In particular s, t are in a common simplex and hence commute. Since $c(s) = c(t)$, we see that $s = t$. Since $q(w \times y) = q(gu \times z)$, we have $w^{-1}gu \in \Gamma_\sigma$ for some simplex σ with $s \in \sigma$. Thus $w^{-1}gu \in Z_s(\Gamma)$ which implies $g \in Z_s(\Gamma)$ and $gM_s = M_s$.

□

Lemma 2.9 corresponds to the following fact of the decomposition $\Gamma = H_s \cup sH_s$.

Lemma 2.10. $c(s) = c(t) \Rightarrow \gamma\partial H_s \subset H_t$ *or* $\gamma\partial H_s \subset tH_t$

Proof. Assume first that $\gamma \in H_t$. Let $a \in Z_s^0(S)$. If $\gamma a \in tH_t$ then by Lemma 2.6 (b) $\ell(\gamma a) > \ell(\gamma)$. Let $s_1 \dots s_k$ be a reduced representation of γ, then γa is reduced. Would $ts_1 \dots s_k a$ be not reduced then by Lemma $t = a$, i.e. $t \in Z_a^0(S)$ and $c(t) \neq c(s)$ a contradiction. Thus $ts_1 \dots s_k a$ is reduced and hence $\ell(t\gamma a) > \ell(\gamma a)$ and hence $\gamma a \in H_t$. By induction $\gamma\alpha \in H_t$ for $\alpha \in Z_s^0(\Gamma)$ hence $\gamma(\partial H_s) \subset H_t$. If $\gamma \in tH_t$ then $t\gamma \in H_t$ and by the above $t\gamma\partial H_s \subset H_t$ which implies $\gamma\partial H_s \subset tH_t$.

□

3 Maps into Trees

In this section we study maps of the Cayley graph into products of trees.

3.1 Components of the Davis Complex

The mirror M_s devides the Davis complex into two connected components corresponding to the decomposition $\Gamma = H_s \cup sH_s$. We have

$$X = \Big(\bigcup_{\gamma\in H_s} \gamma C\Big) \cup \Big(\bigcup_{\gamma\in sH_s} \gamma C\Big)$$

and the common boundary of this parts is M_s. The two parts (and also the common wall) are connected by Lemma 2.6 (d).

If we consider the Cayley graph $C(\Gamma, S)$ as the 1-skeleton $X^{(1)}$ of the Davis complex, then every edge e of the Cayley graph intersects exactly one wall, the edge $[\gamma, \gamma s]$ intersects the wall γM_s. Thus an edge e with colour $c(e)$ of the Cayley graph intersects a mirror with the same colour. We say that two edges e, e' of the Cayley graph are *parallel*, if e and e' intersect the same mirror.

Proof. (of Theorem 1.1)

Let c be a colour. Consider the graph T_c with vertices the connected components of

$$X \setminus \bigcup_{c(s)=c, \gamma \in \Gamma} \gamma M_s$$

and edges correspond to the walls between components. Since the Davis complex is simply connected [D] and every wall devides X in exactly two pieces and different walls of the same color do not intersect, T_c is indeed a tree (see [DJ]). We define a map $p_c : X^{(0)} \to T_c$ by the rule: $p_c(v)$ is the component that contains v. This map extends simplicially to the Cayley graph $X^{(1)}$.

Note that p_c is equivariant. The maps p_c define an equivariant map $\mu : C(\Gamma, S) \to \prod_{c=1}^{n} T_c$. Remember that we take the l_1-metric on $\prod_{c=1}^{n} T_c$. Since every edge in the Cayley graph intersects exactly one mirror, the distance between γ_1 and γ_2 in Γ equals the number of walls between points $q(\gamma_1 \times x_0)$ and $q(\gamma_2 \times x_0)$ where x_0 is the cone point in the chamber C. On the other hand this number is exactly the l_1-distance in the product of our trees. Thus, μ is an isometry. $\square$

3.2 Locally Compact Trees

In section 3.1 we constucted a tree from the connected components of the Davis complex. The edges of the c-tree correspond to the c-mirrors or equivalently to the parallelclasses of c-edges of the Cayley graph. In this section we construct maps into locally compact trees. In a certain sense the following construction is a modification of the result in section 3.1. We have however to reformulate the result in a different language:

We first introduce a general class of rooted simplicial trees: Let $(Q) = Q_1, Q_2, \ldots$ be a sequence of nonempty sets. We associate to (Q) a rooted simplicial tree $T_{(Q)}$ in the following way (compare [dH] p. 211): The set of vertices is the set of finite sequences $(q_1, \ldots, q_k)$ with $q_\kappa \in Q_\kappa$. The empty sequence defines the root vertex and is denoted by $v_\emptyset$. We write the vertex given by $(q_1, \ldots, q_k)$ also as $v_{(q_1, \ldots, q_k)}$. Two vertices are connected by an edge in $T_{(Q)}$ if their length (as sequences) differ by one and the shorter can be obtained by erasing the last term of the longer. The root vertex has $\mid Q_1 \mid$

neighbours and every vertex of distance i to $v_\emptyset$ has $\mid Q_{i+1} + 1 \mid$ neighbours, one ancestor and $\mid Q_{i+1} \mid$ descendents. Here $\mid Q_i \mid$ denotes the cardinality of Q_i. The tree $T_{(Q)}$ is locally compact, if and only if Q_i is finite for all i.

We recollect and extend certain notations. We have given a colouring $c : S \to \{1, \ldots, n\}$ such that $c(s) \neq c(t)$ if $ts = st$ and $t \neq s$. We have the length function $\ell : \Gamma \to \mathbb{N}$ and we introduce for a colour c also the function $\ell_c : \Gamma \to \mathbb{N}$ by defining $\ell_c(\gamma)$ to be the number of letters with colour c in a reduced representation of γ. This does not depend on the special representation. We denote by $\mathcal{R} \subset \Gamma$ the set of all reflections. An element $r \in \mathcal{R}$ can be represented as $r = \gamma s \gamma^{-1}$ for some $s \in S$ and some $\gamma \in \Gamma$. One easily checks that $\gamma_1 s \gamma_1^{-1} = \gamma_2 t \gamma_2^{-1}$ for $\gamma_i \in \Gamma$ and $s, t \in S$ if and only if $s = t$ and $\gamma_1^{-1}\gamma_2 \in Z_s(\Gamma)$. This implies that the maps $g : \mathcal{R} \to S$, $\gamma s \gamma^{-1} \mapsto s$ and $c : \mathcal{R} \to \{1, \ldots, n\}$, $c(r) = c(g(r))$ are well defined.

As defined above the mirror M_r of a reflection $r \in \mathcal{R}$ is a subset of the Davis complex. It intersects the Cayley graph $C(\Gamma, S)$ (viewed as embedded in the Davis complex) in a set of parallel edges: A given edge $e = [\gamma, \gamma s]$ of the Cayley graph intersects the mirror M_r of the reflection $r(e) = \gamma s \gamma^{-1} \in \mathcal{R}$. If on the other side a reflection $r \in \mathcal{R}$ can be represented as $r = \gamma s \gamma^{-1}$ then r interchanges γ and γs and hence M_r intersects the edge of $C(\Gamma, S)$ with endpoints γ and γs. By replacing γ by γs if necessary we can assume that $\ell(\gamma s) > \ell(\gamma)$ and hence we can write this edge as the oriented edge $[\gamma, \gamma s]$. We define the level function $\mathrm{lev} : \mathcal{R} \to \mathbb{N}$ by $\mathrm{lev}(r) = \ell_c(\gamma) + 1$, where $c = c(r)$ and γ is an element such that $r = \gamma s \gamma^{-1}$ and $\ell(\gamma s) > \ell(\gamma)$. It is not difficult to show that the level function is well defined. The level function can also be considered as a function defined on the set of edges of the Cayley graph. This function has an easy geometric interpretation: Let $r \in \mathcal{R}$ with $c = c(r)$. Consider a shortest path in the Cayley graph from the origin 1 to the mirror M_r. Then $\mathrm{lev}(r)$ is the number of mirrors with colour c which are intersected by this path (including the final mirror M_r).

We denote by $\mathcal{R}^c \subset \mathcal{R}$ the reflections with colour c, and by $\mathcal{R}^c_i$ the set of reflections with colour c and level i. We consider the tree $T_{(\mathcal{R}^c)}$ belonging to the sequence $(\mathcal{R}^c) = \mathcal{R}^c_1, \mathcal{R}^c_2, \ldots$.

For a colour c we define a map $\varphi_c : \Gamma \to T_{(\mathcal{R}^c)}$ in the following way: Let $\gamma \in \Gamma$ and let $s_1 \cdots s_k$ be a reduced word representing γ. This corresponds to a geodesic path following the edges $e_1, \ldots, e_k$ in the Cayley graph, where $e_i = [s_1 \cdots s_i, s_1 \cdots s_{i+1}]$. Let $1 \leq i_1 < \ldots < i_{\ell_c(\gamma)} \leq k$ be the set of indices with $c(s_{i_\kappa}) = c(e_{i_\kappa}) = c$. Define

$$r_\kappa = r(e_{i_\kappa}) = s_1 \cdots s_{i_\kappa - 1} s_{i_\kappa} (s_1 \cdots s_{i_\kappa - 1})^{-1} \in \mathcal{R}^c$$

and

$$\varphi_c(\gamma) = (r_1, \ldots, r_{\ell_c(\gamma)})$$

By construction $\mathrm{lev}(r_\kappa) = \kappa$ and $\varphi_c(\gamma) \in T_{(\mathcal{R}^c)}$.

We show that φ_c is well defined. By the deletion rule we have to check the following: let w_1stw_2 and w_1tsw_2 are reduced representations of the same element γ, then the above definition gives the same φ_c-image for both words. Note that $s \neq t$ and s commutes with t. Thus s and t have different colours. If s and t both do not have the colour c then the sequence $(r_1, \ldots, r_{\ell_c(\gamma)})$ does not change at all. If one of s or t has colour c we assume w.l.o.g. that $c(t) = c$ and $c(s) \neq c$. Let this t be the κ-th letter of colour c in the word w_1stw_2, then it is also the κ-th letter of colour c in the word w_1tsw_2. Clearly the reflections $r_1, \ldots, r_{\kappa-1}, r_{\kappa+1}, \ldots, r_{\ell_c(\gamma)}$ are the same for the two words. Since $r_\kappa = w_1st(w_1s)^{-1} = w_1tw_1^{-1}$ we see that r_κ is also the same in both words we are done.

Remark 3.1. The tree T_c from section 3.1 is canonically isometrically embedded into $T_{(\mathcal{R}^c)}$ by the following map χ: The vertices of T_c are the components of the Davis complex minus the c-walls. The component containing the base Chamber C is mapped by χ to the root vertex $v_\emptyset \in T_{(\mathcal{R}^c)}$. If C' is some other component, consider a shortest path from C to C'. This path intersects a sequence of c-walls $M_{r_1}, \ldots, M_{r_k}$. Now define $\chi(C') = v_{(r_1,\ldots,r_k)}$. Clearly the image of φ_c is contained in $\chi(T_c)$. By identifying T_c with its image we obtain a map $\varphi_c : \Gamma \to T_c \subset T_{(\mathcal{R}^c)}$. This map is exactly the map p_c defined in section 3.1.

The sets $\mathcal{R}_i^c$ are (besides of trivial cases) not finite and hence the corresponding tree is not locally compact. In order to get a map into locally compact trees we have to replace the infinite sets $\mathcal{R}_i^c$ by certain finite sets. Therefore we will construct maps $\mathrm{fin}_i^c : \mathcal{R}_i^c \to F_i^c$ where F_i^c are finite sets. These maps have to satisfy the condition that two different reflections whose mirrors are close to each other are mapped to different points. The distance between mirrors is defined as follows

Definition 3.2. *Let M_{r_1} and M_{r_2} be two different mirrors with the same colour $c = c(r_1) = c(r_2)$. Then the distance is defined to be the number*

$$d(M_{r_1}, M_{r_2}) = \inf\{d(\gamma_1, \gamma_2) + 1 \mid \gamma_i \in \Gamma, ,\ s_i \in S,\ r_i = \gamma_i s_i \gamma_i^{-1}\}$$

Remark 3.3. We can view the mirrors M_{r_i} as subsets of the Cayley graph such that M_{r_i} is the set of midpoints of the edges with endpoints γ_i and $\gamma_i s_i$ where $\gamma_i s_i \gamma_i^{-1}$ is a representation of r_i. Since by Lemma 2.9 two different mirrors with the same colour do not intersect, the distance defined above is exactly the distance of the mirrors considered as subsets of the Cayley graph.

The following result is essential for our construction

Proposition 3.4. *There exists a map $\mathrm{fin}_i^c : \mathcal{R}_i^c \to F_i^c$ where F_i^c is a finite set such that $\mathrm{fin}_i^c(r_1) = \mathrm{fin}_i^c(r_2)$ implies $r_1 = r_2$ or $d(M_{r_1}, M_{r_2}) \geq 4ni$. Furthermore there exists a constant $\rho > 0$ (independent of i) such that $|F_i^c| \leq e^{\rho i}$.*

In order to prove Proposition 3.4 we need

Lemma 3.5. *Let $\nu \in \mathbb{N}$ be given. Then there exists a finite group F_ν and a homomorphisms $\sigma : \Gamma \to F_\nu$ with the property: If $r_1, r_2 \in \mathcal{R}$ with $g(r_1) = g(r_2)$, $d(M_{r_1}, M_{r_2}) < \nu$ and $\sigma(r_1) = \sigma(r_2)$ then $r_1 = r_2$. Furthermore there exists a constant ρ_1 (independent of ν) such that $| F_\nu | \leq e^{\rho_1 \nu}$.*

Proof. Let $\alpha_1, \dots, \alpha_k \in \Gamma$ be the set of nontrivial elements with $\ell(\alpha_i) \leq 2\nu + 2$. Since a Coxeter group Γ is residually finite, there exists a finite group F_ν and a homomorphism $\sigma : \Gamma \to F_\nu$ such that $\sigma(\alpha_i) \neq 1$ for all $i = 1, \dots, k$. Let $r_1, r_2 \in \mathcal{R}$ with $g(r_1) = g(r_2)$ and $d(M_{r_1}, M_{r_2}) < \nu$. Then there exits $s \in S$ and $\gamma_j \in \Gamma$ such that $r_j = \gamma_j s \gamma_j^{-1}$ and $d(\gamma_1, \gamma_2) \leq \nu$. Let $\tau = \gamma_1^{-1}\gamma_2$, hence $\ell(\tau) \leq \nu$. By assumption $r_1^{-1} r_2 \in ker(\sigma)$. Since $r_1^{-1} r_2 = \gamma_1 s \tau s \tau^{-1} \gamma_1^{-1}$ we have $s\tau s\tau^{-1} \in ker(\sigma)$. Since $\ell(s\tau s \tau^{-1}) \leq 2\nu + 2$ we have by construction that $s\tau s\tau^{-1}$ is trivial and hence τ commutes with s which implies that $r_1 = r_2$.

We finally have to estimate the size of F_ν. Therefore we use a concrete geometric realisation of Γ as a group of linear transformations on a vectorspace V over $\mathbb{R}$ (compare [H] p. 108). V has the basis $\{v_s \mid s \in S\}$ in one to one correspondence to S. For each $s \in S$ define a reflection $h(s) : V \to V$ by $h(s)(v_s) = -v_s$, $h(s)(v_t) = v_t + 2v_s$ if $t \neq s$ and $ts \neq st$, $h(s)(v_t) = v_t - 2v_s$ if $t \neq s$ and $ts = st$. Then h defines a faithful representation of Γ and enables us to identify Γ with a subgroup of $GL(| S |, \mathbb{Z})$. The description also implies that every matrix coefficient of $h(\alpha)$ is an integer with norm $\leq 3^{\ell(\alpha)}$. Let $F_\nu = GL(| S |, \mathbb{Z}/(3^{2\nu+2} + 1)\mathbb{Z})$, then the canonical map $\sigma : \Gamma \to F_\nu$ satisfies $\sigma(\alpha) \neq 1$ if $0 < \ell(\alpha) \leq 2\nu + 2$. Clearly $| F_\nu | \leq e^{\rho_1 \nu}$ where ρ_1 only depends on $| S |$.

□

Proof. (of Proposition 3.4)

Let $c \in \{1, \dots, n\}$ be a given colour and $i \in \{1, 2, \dots\}$ be a given level. Let $S^c \subset S$ be the subset of generators with colour c. For $s \in S^c$ let $\mathcal{R}_i^s$ be the set of reflections with generator s and level i, $\mathcal{R}_i^c = \bigcup_{s \in S^c} \mathcal{R}_i^s$. By Lemma 3.5 there exists a set F_i^s and a map $\mathrm{fin}_i^s : \mathcal{R}_i^s \to F_i^s$ with the property: if $\mathrm{fin}_i^s(r_1) = \mathrm{fin}_i^s(r_2)$ then $r_1 = r_2$ or $d(M_{r_1}, M_{r_2}) \geq 4ni$. Define fin_i^c to be the sum of the maps fin_i^s , $s \in S^c$ from $\mathcal{R}_i^c$ to F_i^c.

□

Remark 3.6. For simplicity we use the notation fin instead of fin_i^c if the indices are clear from the context.

For a colour c we consider the locally compact tree $T_{(F^c)}$ coming from the sequence $(F^c) = F_1^c, F_2^c, \cdots$. We consider the map $\psi_c : \Gamma \to T_{(F^c)}$ defined by

$$\psi_c(\gamma) = (\mathrm{fin}(r_1), \dots, \mathrm{fin}(r_{\ell_c(\gamma)}))$$

where

$$\varphi_c(\gamma) = (r_1, \ldots, r_{\ell_c(\gamma)})$$

This map extends naturally to a map

$$\psi_c : C(\Gamma, S) \to T_{(F^c)}$$

We define

$$\psi = \prod_{c=1}^{n} \psi_c : C(\Gamma, S) \to \prod_{c=1}^{n} T_{(F^c)}$$

Essential for the bilipschitz property of the map ψ is the following result

Lemma 3.7. *Let $\gamma, \gamma' \in \Gamma$ and let $d(\psi(\gamma), \psi(\gamma')) = m$ then $d(\gamma, \gamma') \leq 16nm$.*

Proof. By Lemma 2.2 there are exists an element $\alpha \in \Gamma$ between γ and γ' such that α lies also between 1 and γ and 1 and γ'. We now consider a geodesic $\alpha = \alpha_0, \cdots, \alpha_\tau = \gamma$ from α to γ and a geodesic $\alpha = \alpha'_0, \cdots, \alpha'_{\tau'} = \gamma'$ from α to γ'. Then $\tau + \tau' = d(\gamma, \gamma')$. The edges $e_i = [\alpha_{i-1}, \alpha_i]$ and $e_{i'} = [\alpha'_{i-1}, \alpha'_i]$ are then oriented edges of the Cayley graph and the path $e_\tau, \ldots, e_1, e'_1, \ldots, e'_{\tau'}$ is a geodesic in $C(\Gamma, S)$ from γ to γ' with length $\tau + \tau' = d(\gamma, \gamma')$. We can assume without loss of generality that $\tau \geq \tau'$. Let τ_c be the number of c-edges in the path $e_1, \ldots, e_r$. Choose the colour $c \in \{1, \ldots, n\}$ in a way that τ_c is maximal. If $\tau_c \leq 8m$ then $d(\gamma, \gamma') = \tau + \tau' \leq 2n\tau_c \leq 16nm$ and we are done.

Thus we can assume $\tau_c > 8m$. Let e_j be the $(\tau_c - m)$-th c-edge in the geodesic path $e_1, \ldots, e_\tau$ and let r_j be the corresponding reflection. Since $m = d(\psi(\gamma), \psi(\gamma')) \geq d(\psi_c(\gamma), \psi_c(\gamma'))$ there exists an c-edge e' in the path $e'_1, \ldots, e'_{\tau'}$ with corresponding reflection r' such that $\mathrm{lev}(r') = \mathrm{lev}(r_j)$ and $\mathrm{fin}(r') = \mathrm{fin}(r_j)$. We claim that $r_j = r'$. Note that $\mathrm{lev}(r_j) \geq (\tau_c - m)$ and $d(M_{r_j}, M_{r'}) < \tau + \tau' \leq 2n\tau_c$. If $r_j \neq r'$, then by Proposition 3.4 we have

$$d(M_{r_j}, M_{r'}) \geq 4n \,\mathrm{lev}(r_j)) \geq 4n(\tau_c - m) \geq 4n(\tau_c - \frac{\tau_c}{8}) = 2n\tau_c$$

a contradiction.

Thus $r_j = r' = \beta s \beta^{-1}$ for some $\beta \in \Gamma$ and some $s \in S$. The edges $e_j = [\alpha_{j-1}, \alpha_{j-1}s]$ and $e' = [\alpha'_{j'-1}, \alpha'_{j'-1}s]$ are parallel and intersect the same mirror βM_s. Thus α_{j-1} and $\alpha'_{j'-1}$ are both in βH_s and hence by Lemma 2.6 the geodesic $\alpha_{j-1}, \ldots, \alpha_0 = \alpha'_0, \ldots, \alpha'_{j'-1}$ is completely contained in βH_s and in particular $\alpha \in \beta H_s$. Now α_j and $\alpha'_{j'}$ are contained in $\beta s H_s$ and by the same argument the complete geodesic $\alpha_j, \ldots, \alpha_0 = \alpha'_0, \ldots, \alpha'_{j'}$ is contained in $\beta s H_s$. Hence $\alpha \in \beta H_s \cap \beta s H_s = \emptyset$. Thus the assuption $\tau_c > 8m$ leads to a contradiction. $\square$

Proof. (of Theorem 1.3)

The map $\psi : C(\Gamma, S) \to T_{(F^c)}$ is clearly 1-lipschitz. By Lemma 3.7 ψ is bilipschitz. Let T be any exponentially branching tree. Since the tree $T_{(F^c)}$ satisfies the estimate $\mid F_i^c \mid \leq e^{\rho i}$, one can show that there exists a bilipschitz embedding of $T_{(F^c)}$ into T (we do *not* require that the root vertex of $T_{(F^c)}$ is mapped to a given basevertex $t_0 \in T$). Combining these results we obtain a bilipschitz embedding of $C(\Gamma, S)$ into $T_{(F^c)}$. Since the maps $\mathrm{fin}_i^c : (\mathcal{R}_i^c, d) \to F_i^s$ where $d(R_1, r_2) = d(M_{r_1}, M_{r_2})$ are locally injective by 3.4, the map ψ has locally the same properties as the map μ, i.e. it is locally an isometry. By adjusting the constants in 3.4 suitable, we can enforce that ψ is an isometric embedding on every ball of a given radius r. □

Proof. (of Corollary 1.4)

Consider the right angled Coxeter group Γ given by the generator set $S = \{s_1, \dots, s_6\}$ and relations $s_i s_{i+1} = s_{i+1} s_i$ (indices mod 6). This group acts discretely on the hyperbolic plane such that a Dirichlet fundamental domain is bounded by the regular right angled hexagon in $\mathbb{H}^2$. By Theorem 1.3. we can embed the Cayley graph of Γ locally isometric and globally bilipschitz into a product of trees. The Cayley graph can be realized canonically in $\mathbb{H}^2$, where the vertices are the central points of the hexagonal decomposition of $\mathbb{H}^2$ and the edges are geodesics. Then the Cayley graph gives a decomposition of $\mathbb{H}^2$ into regular quadrilaterals (with angles equal to $\frac{\pi}{3}$). The map ψ maps the boundary of a square to a square in $T \times T$. Thus the maps ψ can be extended from the Cayley graph in a bilipschitz way to all of $\mathbb{H}^2$. □

Proof. (of Corollary 1.5)

By a result of Brady and Farb [BF] there exists a bilipschitz embedding of the hyperbolic space $\mathbb{H}^n$ into the $(n-1)$-fold product of hyperbolic planes. Actually in [BF] it is only stated that the embedding is quasiisometric but their proof gives a bilipschitz embedding (compare [F, section 2]). Combining Corollary 1.4 with this result, we are done.

□

Proof. (of Corollary 1.6)

A recent result of Januszkiewicz and Swiatkowski [JS] shows the existence of a Gromov-hyperbolic right angled Coxeter group Γ_n of arbitrary given n, such that the virtual cohomological dimension of Γ_n is n. The construction of these groups imply that they have chromatic number n. By Theorem 1.3 Γ_n can be embedded in a bilipschitz way into the product T^n. □

Proof. (of Corollary 1.7)

We recall the definition of the hyperbolic rank of a metric space. Given a metric space M consider all locally compact Gromov-hyperbolic subspaces

Y quasiisometrically embedded into M. Then $\mathrm{rank}_h(M) = \sup_Y \dim \partial_\infty Y$ is called the hyperbolic rank. (Compare [BS1] for a discussion of this notion). Let T be an exponentially branching tree. Let Γ_n as in the proof of Corollary 1.6 above. Γ_n can be embedded in a bilipschitz way into the product T^n. Since the virtual cohomological dimension of Γ_n is n, we have $\dim \partial_\infty \Gamma_n = (n-1)$ by [BM]. Thus $\mathrm{rank}_h(T^n) \geq (n-1)$ by the definition of the hyperbolic rank. The opposite inequality $\mathrm{rank}_h(T^n) \leq (n-1)$ follows from standard topological considerations.

□

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